QP II

Final exam
2025

Assignment date: July 5th, 2024, 15h15
Due date: July 5th, 2024, 18h15

PHYS-314 – Exam – room PO 01

- You must answer ALL questions in the short answer section.
- You must answer precisely 2 (out of 3) of the questions in the long answer section.

  Please mark clearly which two you have answered below and **start a new sheet for each of the long answer questions**.
- **Write your solutions in the indicated space.** Scrap paper will not be corrected.
- You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.
- A simple calculator (without internet access) is allowed.
- Please write your name on the top right corner of each sheet you use.
- Good luck! Enjoy!

NAME STICKER GOES HERE

Short answers:	/ 50
Problem A: YES or NO	/ 25
Problem B: YES or NO	/ 25
Problem C: YES or NO	/ 25
Total	/100

Short questions

1. **Bloch Sphere**

   Density matrices satisfy the following 3 conditions:

   - The density matrix is Hermitian: $\hat{\rho}^\dagger = \hat{\rho}$
   - It has trace 1: $\text{Tr}\hat{\rho} = 1$
   - It is positive or null : $\langle\Psi|\hat{\rho}|\Psi\rangle \geq 0, \quad \forall\Psi$

   a) Show that any density matrix $\hat{\rho}$ of a 2 level system can be written

   $$\hat{\rho} = \frac{1}{2}(\hat{I} + \hat{\boldsymbol{\sigma}}.\boldsymbol{r}), \tag{1}$$

   where $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$. Argue that $\boldsymbol{r}$ is a real vector of 3D space and $|\boldsymbol{r}| \leq 1$.

   Hint: The eigenvalues of $\hat{\rho}$ are $\frac{1}{2}(1 \pm |\boldsymbol{r}|)$.

   (3 marks)

   b) A pure state is a density operator that can be written in the form $\hat{\rho} = |\psi\rangle\langle\psi|$. Show that the Bloch vector $\boldsymbol{r}$ for a pure state has norm 1, $|\boldsymbol{r}| = 1$.

   (4 marks)

   c) A mixed state is a density operator that is a convex combination of pure states. That is,

   $$\hat{\rho}_{\text{mixed}} = \sum_i p_i|\psi_i\rangle\langle\psi_i|$$

   with $0 < p_i < 1$ and $\sum_i p_i = 1$.

   Based on this definition of a mixed state give a geometric argument to show that the Bloch vector of a mixed state is always less than 1, $|\boldsymbol{r}_{\text{mixed}}| < 1$.

   (3 marks)

2. **Entanglement**

   Alice, Bob and Charlie share a three qubit state:

   $$|\psi\rangle_{ABC} = \frac{1}{\sqrt{2}}\left(|000\rangle_{ABC} + |111\rangle_{ABC}\right) \tag{2}$$

   where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle$ are the ± 1 eigenstates of $\hat{\sigma}_x$ and $|0\rangle$ and $|1\rangle$ are the ± 1 eigenstates of $\hat{\sigma}_z$ respectively.

   a) Suppose Bob measures his qubit in the $\hat{\sigma}_z$ basis. Find the output states for Alice and Charlie conditional on Bob's measuring 0 and conditional on Bob measuring 1.

   (2 marks)

b) What if Bob instead measures in the $\hat{\sigma}_x$ basis? State the output states for Alice and Charlie conditional on Bob's measurement outcomes in this case.

(3 marks)

c) Assuming Bob does not tell Alice or Charlie the output to his measurement, what is the mixed state that Alice and Charlie share after his measurement in the $\hat{\sigma}_z$ basis?

(2 marks)

d) Could Alice and Charlie use this state to violate a Bell inequality? Explain your answer.

(3 marks)

3. **Fermions and Bosons**

Which of the following states are valid Fermionic states and which are valid Bosonic states:

$$\text{(a)} \quad \left(\frac{1}{\sqrt{2}}|p\rangle_1|q\rangle_2 + \frac{1}{\sqrt{2}}|q\rangle_1|p\rangle_2\right)\left(\frac{1}{\sqrt{2}}|R\rangle_1|L\rangle_2 - \frac{1}{\sqrt{2}}|L\rangle_1|R\rangle_2\right) \tag{3}$$

$$\text{(b)} \quad \left(\frac{1}{\sqrt{2}}|p\rangle_1|q\rangle_2 - \frac{1}{\sqrt{2}}|q\rangle_1|p\rangle_2\right)|R\rangle_1|R\rangle_2 \tag{4}$$

$$\text{(c)} \quad \left(\frac{1}{\sqrt{2}}|p\rangle_1|q\rangle_2 + \frac{1}{\sqrt{2}}|q\rangle_1|p\rangle_2\right)\left(\frac{1}{\sqrt{2}}|R\rangle_1|L\rangle_2 + \frac{1}{\sqrt{2}}|L\rangle_1|R\rangle_2\right) \tag{5}$$

$$\text{(d)} \quad |p\rangle_1|q\rangle_2|R\rangle_1|L\rangle_2 \tag{6}$$

$$\text{(e)} \quad \left(\frac{1}{\sqrt{2}}|p\rangle_1|q\rangle_2 - \frac{1}{\sqrt{2}}|q\rangle_1|p\rangle_2\right)\left(\frac{1}{\sqrt{2}}|R\rangle_1|L\rangle_2 - \frac{1}{\sqrt{2}}|L\rangle_1|R\rangle_2\right) \tag{7}$$

(5 marks)

4. **Symmetry.** Consider a unitary irreducible representation $R(g) = U_g$ of group G.

a) Use the Grand Orthogonality Theorem to prove that

$$\frac{1}{N}\sum_g U_g X U_g^\dagger = \frac{1}{d}\,\mathrm{Tr}[X]\,I \tag{8}$$

where X is an arbitrary operator, $d = \dim(X)$ and N is the order of the group.

(4 marks)

The above relation for averaging over irreducible representations of finite groups generalizes to averaging over compact Lie groups. In this case the finite average $\frac{1}{N}\sum_g$ becomes a continuous integral over a uniform measure $\int d\mu(g)$ and we have:

$$\langle X\rangle_G := \int_G d\mu(g) U_g X U_g^\dagger = \frac{1}{d}\,\mathrm{Tr}[X]\,I \tag{9}$$

b) Use this result to explain why applying random single qubit rotations to any single qubit state on average results in the maximally mixed state.

(3 marks)

c) We now consider only random rotations about the x-axis. What is the relevant symmetry group and representation in this case? Why can Eq. (9) not be directly applied to compute this average?

(3 marks)

d) Explain how a different version of Eq. (9) *can* be applied to compute the state that results on average from applying a random x rotation to a qubit.

(4 mark)

e) Hence compute the state that results on average from applying a random x rotation to a qubit.

(2 marks)

5. **Variational Principle.** Consider the 1D Harmonic Oscillator with H:

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2$$

Use the variational principle with the trial wavefunction

$$\psi(x) = Ae^{-bx^2}$$

to upper bound the ground state energy of H, where $A = (2b/\pi)^{1/4}$ is the normalization constant.

(9 marks)

You may find the following integral helpful:

$$\int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \frac{\sqrt{\pi}}{2a^{3/2}}. \tag{10}$$

Longer questions

Please **pick 2 questions** to attempt - mark your choices clearly on the cover sheet.

Start a new sheet for each question.

Question A - Perturbation Theory

Consider a free particle in a box of width a, with sides at $x = 0$ and $x = a$. The unperturbed problem is well known: the eigenvalues are

$$E_n^{(0)} = n^2\hbar^2\pi^2/2ma^2 = n^2 E_1^{(0)},$$

and the eigenfunctions are

$$\langle x|n_0\rangle = u_n(x) = \sqrt{2/a}\,\sin(n\pi x/a).$$

(a) We now add a perturbation

$$H_1 = W\cos(\pi x/a).$$

Sketch the perturbed potential well as a function of x. Show that all the first-order energy shifts are zero.

(4 marks)

(b) Find the first order correction to the ground state wavefunction. Sketch the ground state wavefunction and the correction.

(7 marks)

(c) What constraints are required on W for perturbation theory to be a suitable approximation method?

(3 marks)

(d) What is the second-order shift $E_n^{(2)}$ for $n = 1$ and $n = 2$?

(11 marks)

Hint: You will find the evaluation of the integrals much simplified if you start by proving for the perturbation a relationship of the form

$$H_1 u_n = \alpha(u_{n-1} + u_{n+1}).$$

This relationship turns the integrals into orthogonality integrals. You will need to think about the meaning of this equation for $n = 1$ since $n - 1$ is then zero, while u_n is only defined for $n > 0$.

Question B - Symmetry

The quaternion group Q_8 is an order 8 non-abelian group which is isomorphic to the quaternions under multiplication. Do not worry if you do not know anything about quaternions, all you need to know to address this question is that Q_8 has the following Cayley table:

	e	$\bar{e}$	i	$\bar{i}$	j	$\bar{j}$	k	$\bar{k}$
e	e	$\bar{e}$	i	$\bar{i}$	j	$\bar{j}$	k	$\bar{k}$
$\bar{e}$	$\bar{e}$	e	$\bar{i}$	i	$\bar{j}$	j	$\bar{k}$	k
i	i	$\bar{i}$	$\bar{e}$	e	k	$\bar{k}$	$\bar{j}$	j
$\bar{i}$	$\bar{i}$	i	e	$\bar{e}$	$\bar{k}$	k	j	$\bar{j}$
j	j	$\bar{j}$	$\bar{k}$	k	$\bar{e}$	e	i	$\bar{i}$
$\bar{j}$	$\bar{j}$	j	k	$\bar{k}$	e	$\bar{e}$	$\bar{i}$	i
k	k	$\bar{k}$	j	$\bar{j}$	$\bar{i}$	i	$\bar{e}$	e
$\bar{k}$	$\bar{k}$	k	$\bar{j}$	j	i	$\bar{i}$	e	$\bar{e}$

1. What order are the proper (i.e., non-trivial) subgroups of Q_8?

   (2 marks)

2. Find two of the proper subgroups of Q_8.

   (3 marks)

3. The conjugacy classes of Q_8 are

   $$\{e\}, \{\bar{e}\}, \{i, \bar{i}\}, \{j, \bar{j}\}, \{k, \bar{k}\}$$

   Verify that $\{e\}$ and $\{k, \bar{k}\}$ are indeed conjugacy classes.

   (3 marks)

4. What is an irreducible representation (or, 'irrep' for short)? How many (non-equivalent) irreducible representations does Q_8 have?

   (3 marks)

5. Consider this representation of Q_8:

$$e \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \bar{e} \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$i \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad \bar{i} \mapsto \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \bar{j} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$k \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad \bar{k} \mapsto \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

State a theorem that allows you to determine whether a representation is irreducible. Hence determine whether this representation is irreducible.

(3 marks)

6. State a theorem that allows you to determine the dimensions of a groups irreps. Hence, what are the dimensions of each of the quarternion's groups irreps?

   (3 marks)

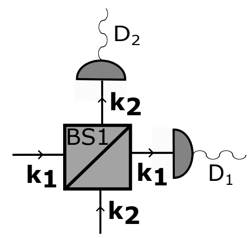
7. State a theorem that can help identify a groups 1D irreps. Hence identify the quaternion's groups 1D irreps.

   (Hint you will need to use the Cayley table to help find the irreps. You may also find that recalling the irreps of C3v is helpful to guess.)

   (8 marks)

Question C - Quantum Bomb Testing

Let's start by getting familiar with a parameterized beamsplitter of the form sketched below:



The action of this parameterized beamsplitter on the mode operators $(a_1^\dagger, a_2^\dagger)$ is given by the unitary

$$U_{BS1} = U_{BS2}^\dagger = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

1. Find the output state for the case that the initial state contains: (i) 1 photon in mode $\mathbf{k}_1$, vacuum in mode $\mathbf{k}_2$ (ii) One photon in each mode (iii) Two photons in mode $\mathbf{k}_1$.

   (10 marks)

2. Suppose you place photon detectors in both output modes of the beamsplitter. Would it be possible to determine which of the three initial states, (i) (ii) or (iii), you started with after a single run of the experiment? Explain.

   (2 marks)

3. Would it be possible to determine which of the three initial states, (i) (ii) or (iii), you started with after many runs of the experiment?

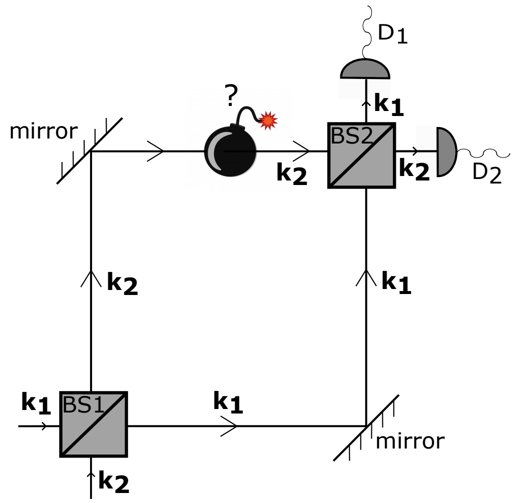
   (3 marks)

The Mach Zender interferometer is a variant on the double slit experiment where there are only two possible paths for a photon to take. Let's consider a version with a parameterized beamsplitter and a box that potentially contains a bomb in one arm as shown in the figure below. Can we use quantum trickery to test whether it contains a live bomb without actually setting the bomb off?

Assume a single photon enters the interferometer through the left hand arm (mode $\mathbf{k_1}$).

4. Assuming there is no bomb in the interferometer, what is the probability of measuring a photon at detectors 1 and 2 respectively?

   (1 marks)



Suppose now there is a bomb in the interferometer. If the bomb does not explode, then the photon is collapsed back into being definitely in the $\mathbf{k}_1$ arm of the interferometer.

There are three possible outcomes when a bomb is in the interferometer:

A) The bomb explodes.

B) The bomb does not explode but you can conclude with certainty that the interferometer does contain a bomb.

C) The bomb does not explode and you cannot tell whether or not there is a bomb.

5. What is the probability of the bomb exploding (i.e. option A) ?

   (3 marks)

6. What is the probability of finding the photon at detectors 1 and 2 if there is a bomb but it does not explode?

   (3 marks)

7. Hence, what are the probabilities of options B (you detect the bomb) and C (you cannot tell whether or not there is a bomb)?

   (3 marks)